Music Temperaments Evaluation Based on Triads

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ABSTRACT

It is impossible for one temperament to achieve optimally both of consonance and modulation. The dissonance level has been calculated by the ratio of two pitch frequencies, however in the current homophonic music, the level should be measured by chords, especially by triads. In this research, we propose to quantify them as Dissonance Index of Triads (DIT). We select eight well-known temperaments and calculate seven diatonic chords in 12 keys and compare the weighted average and standard deviation to quantify the consonance, and then we visualize our experimental results in a two-dimensional chart to compare the trade-offs between consonance and modulation.

1. INTRODUCTION

Nowadays, 12-tone equal temperament is prevalent and other temperaments has fallen to only historical and mathematical interests. However, even now in the actual performance, string and wind instruments are played in an *ad hoc* adjustment of pitches unless accompanied by keyboard instruments. In this age, those electronic instruments ease us in using any scale more freely. Then, our motivation in this paper is to give quantitative understanding to the dissonance level in various temperament in terms of triads.

The difference of temperaments has been often mentioned by the ratio of two pitch frequencies and such web site as *Pianoteq* [1] provides us a very convincing interface to experience the difference of temperaments; however, there were no mathematical formulation to evaluate the consonance and modulation [2] of a chord.

Consonance and dissonance are ambiguous psychological notions. The purpose of this research is to explore the mathematical model of Dissonance Index of Triads (DIT). In 1863, Helmholtz [1] proposed the mathematical model of consonance and dissonance in tones in terms of beats and roughness. In 1965, Plomp and Levelt [2] defined the dissonance curve between two pure tones. Later, the mathematical formula of the curve has been improved, and Vassilakis [3] claimed that the formula he proposed had been believed to be most reliable; and thus, we employ it also in this paper, though adding the effect of overtones.

[1] https://www.pianoteq.com/
[2] In this paper, the modulation means a key transposition.

Ogata [4] has proposed the idea of consonance value among chords, calculating the consonance value of each two tones in the triads and adding them up, and then drew a 3D dissonance curve of the chords. Also, Cook [5] showed the acoustical properties of triads, claiming the perception of harmony is not simply a sum of inner consonance. In this research, we revise the Ogata's calculation and formalize the dissonance level in a more rigorous way.

This paper is organized as follows. In the following section, we show preliminaries including the introduction of various temperaments. Thereafter, we propose our formalization and show its visualization. Then, we analyze the results, and finally conclude.

2. PRELIMINARIES

2.1 Scale and Temperaments

A set of notes employed in a music piece is, when arranged in a pitch order in an octave, is called a scale. The ratio of frequency of two pitches is fixed by natural science disciplines such as physical science, acoustics, and psychology, among which mathematics plays the most important role, and one fixed series of ratios in a scale gives the notion of *temperament*.

Pythagoras in ancient Greece discovered that the perfect fifth interval with the frequency ratio of 3:2 as the most consonant, next to the octave of 2:1, around 550 BC [6]. The *Sanfen Sunyi-fa* by Jing Fang in China (BC77–BC37) is considered to have invented the same temperament with Pythagorean tuning [7].

Since then, musicologists have been constantly exploring how to solve the problem of *Pythagorean comma*, that is the error which slightly exceeds the octave when the 12th tone is introduced by multiples of 3/2. If we peremptorily regard the 12th locates at the octave, the interval between the 11th and the 12th becomes narrower than the other fifths. In later years, Pythagoras pitch was amended to place the narrow fifth, so called the wolf fifth [3], between G♯ and D♯ where the fifth is rarely used.

The ultimate temperament for consonance is the just intonation, introducing the multiple by 5 in addition to 3/2, where the ratio between the intervals can be expressed all by small integers [8]. However, in contrast, the just intonation is very clumsy in modulation. The scale evolved into the mean-tone systems [9, 10], well-temperaments by Andreas Werckmeister (1645–1706) [11] or by Johann Philipp Kirnberger (1721–1783) [12], and since then there exist hundreds or even thousands of music temperaments.

[3] It is named after the unpleasant sound like the roaring of wolves.

	C	^{b}D	D	^{b}E	E	F	$^{\#}F$	G	^{b}A	A	^{b}B	B
Sanfen Sunyi-fa	1	$\frac{3^7}{2^{11}}$	$\frac{9}{8}$	$\frac{3^9}{2^{14}}$	$\frac{81}{64}$	$\frac{3^{11}}{2^{17}}$	$\frac{729}{512}$	$\frac{3}{2}$	$\frac{2^{12}}{3^8}$	$\frac{27}{16}$	$\frac{2^{15}}{3^{10}}$	$\frac{243}{128}$
Pythagorean Tuning	1	$\frac{256}{243}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{729}{512}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{27}{16}$	$\frac{16}{9}$	$\frac{243}{128}$
Just Intonation	1	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$
Quarter-Comma Meantone	1	$\frac{2^3}{5^{\frac{5}{4}}}$	$\frac{5^{\frac{1}{2}}}{2}$	$\frac{2^2}{5^{\frac{3}{4}}}$	$\frac{5}{4}$	$\frac{2}{5^{\frac{1}{4}}}$	$\frac{5^{\frac{6}{4}}}{2^3}$	$5^{\frac{1}{4}}$	$\frac{8}{5}$	$\frac{5^{\frac{3}{4}}}{2}$	$\frac{2^2}{5^{\frac{1}{2}}}$	$\frac{5^{\frac{5}{4}}}{2^2}$
Conventional QC Meantone	1	$\frac{5^{\frac{7}{4}}}{2^4}$	$\frac{5^{\frac{1}{2}}}{2}$	$\frac{2^2}{5^{\frac{3}{4}}}$	$\frac{5}{4}$	$\frac{2}{5^{\frac{1}{4}}}$	$\frac{5^{\frac{6}{4}}}{2^3}$	$5^{\frac{1}{4}}$	$\frac{5^2}{2^4}$	$\frac{5^{\frac{3}{4}}}{2}$	$\frac{2^2}{5^{\frac{1}{2}}}$	$\frac{5^{\frac{5}{4}}}{2^2}$
Werckmeister	1	$\frac{2^8}{3^5}$	$\frac{64\sqrt{2}}{81}$	$\frac{32}{27}$	$\frac{2^{13}\sqrt[4]{2}}{3^8}$	$\frac{4}{3}$	$\frac{2^{10}}{3^6}$	$\frac{8\sqrt[4]{8}}{9}$	$\frac{128}{81}$	$\frac{2^{10}\sqrt[4]{2}}{3^6}$	$\frac{16}{9}$	$\frac{128\sqrt[4]{2}}{81}$
Kirnberger	1	$\frac{135}{128}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{3\sqrt{5}}{4}$	$\frac{16}{9}$	$\frac{15}{8}$
Equal Temperament	1	$2^{\frac{1}{12}}$	$2^{\frac{2}{12}}$	$2^{\frac{3}{12}}$	$2^{\frac{4}{12}}$	$2^{\frac{5}{12}}$	$2^{\frac{6}{12}}$	$2^{\frac{7}{12}}$	$2^{\frac{8}{12}}$	$2^{\frac{9}{12}}$	$2^{\frac{10}{12}}$	$2^{\frac{11}{12}}$

Table 1: Ratios of Temperaments in Fractions

The equal temperament has been the product of compromise, which systematically compensated the Pythagorean comma, defining each half tone to be the 12th root of 2. Then, the temperament perfectly eased the modulation, that is, to enable us to change from one key to another freely, and was applied to the tuning of most modern musical instruments around the world. But we can never say that the equal temperament is satisfactory because it rejects the original intention of the temperament, *viz.*, the consonance between intervals. Table 1 lists the frequency ratios of some typical music temperaments introduced above.

2.2 Helmholtz's Theory of Beats

In physics, the superposition of two simple sinusoidal waves with similar but slightly different frequency will cause periodic fluctuation in strength through time. This phenomenon is known to piano tuners as *beats*. Hermann Helmholtz [1] concluded that dissonance is produced by the beats between two pure tones (without overtones) or between a pair of partials of two complex sounds.

When the difference in frequency is small, the beats can be easily heard. As the difference is increased to 20-30 Hz, the beats will create the impression like 'jarring and rough' described by Helmholtz. Beyond this approximate point, the beats gradually become too rapid to be identified and the sensation of roughness disappears.

2.3 Dissonance Curve

In 1965, Plomp and Levelt confirmed Helmholtz's hypothesis by several experiments [2]. They plotted the dissonance curve and proposed the concept of critical bandwidth. Note that though the sound produced by the musical instruments has a complex timbre this psychological experiment employed only pure tones with the simplest spectrum. The combined experimental results is shown in Figure 1, and nowadays this result is widely accepted.

The figure shows the consonance/dissonance feeling when the frequency is apart from the fixed base tone. The vertical axis on the right side of the figure represents the degree

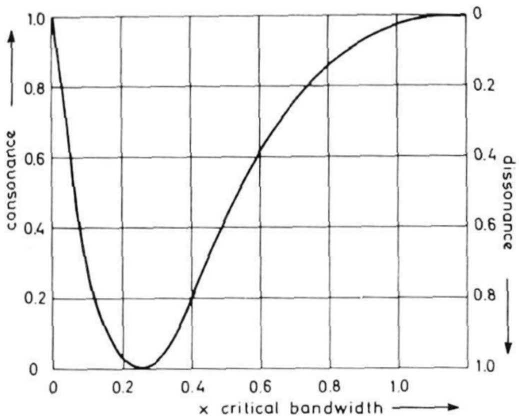

Figure 1: Dissonance Curve From a Fixed Tone to Another Tone [2]

of dissonance, and the interval is from 0 to 1 from top to bottom. The lower the vertical value is in this figure, the more dissonant. The horizontal axis is the frequency difference between higher tone and the base tone, divided by the value of the critical bandwidth. As the frequency difference gradually becomes larger, we can observe the result of the dissonance value d between the two pure tones varying. The most dissonant position $(d = 1)$ is said to be about a quarter of the critical band.

When the frequency of the tone is too high or too low to be heard by human ears, the identification of the tones becomes not that easy. When the horizontal axis of the dissonance curve only takes the frequency difference (without divided by critical bandwidth), we need to draw many different graphs according to the difference of base tones.

2.4 Numerical Calculation of Dissonance in Two Tones

Among various proposals [13–15] on the numerical calculation of dissonance, Vassilakis suggested two principal studies [16, 17], incorporating the notion of roughness [3].

Given a signal whose spectrum has two sinusoidal components with frequencies f_1, f_2 and amplitudes v_1, v_2, where

$$f_{min} = min(f_1, f_2), f_{max} = max(f_1, f_2),$$
$$v_{min} = min(v_1, v_2), v_{max} = max(v_1, v_2),$$

the roughness (dissonance value) of d becomes:

$$d(f_1, f_2, v_1, v_2) = X^{0.1} \cdot 0.5(Y^{3.11}) \cdot Z \quad (1)$$

in which

$$X = v_{min} \cdot v_{max}$$
$$Y = 2v_{min}/(v_{min} + v_{max})$$
$$Z = e^{-b_1 s(f_{max} - f_{min})} - e^{-b_2 s(f_{max} - f_{min})}$$

with $b_1 = 3.5, b_2 = 5.75$,

$$s = \frac{0.24}{s_1 f_{min} + s_2}; s_1 = 0.0207; s_2 = 18.96.$$

Vassilakis has confirmed that his formula reliably and efficiently represents the perception of roughness and performs better than the preceding formulae. Therefore, the temperament evaluation model in this paper is made under this function of dissonance curve.

We generalize the roughness value (1) to include multiple, more than two sinusoidal partials as the sum of each pair of two partials. Suppose a spectrum F with fundamental frequency f is a collection of n sinusoidal waves (or partials) with frequencies $a_1 f, a_2 f, ..., a_n f$ and amplitudes $v_1, v_2, ..., v_n$. Also, we assume that each tone contains n overtones of $[a_1, a_2, ..., a_n] = [1, 2, ..., n]$. According to [4], we also assume that $v_1, v_2, ..., v_n$ is a geometric progression with common ratio of 0.9; that is, $v_1, v_2, ..., v_n = 1, 0.9, 0.81, ..., 0.9^{n-1}$. So when two notes of F_1 and F_2 are played simultaneously, the dissonance value $D(F_1, F_2)$ between them is

$$D(F_1, F_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} d(if_1, jf_2, v_i, v_j) \quad (2)$$

When F_1 and F_2 are at interval t and with the same amplitude (e.g. $F_2 = tF_1$), the transposed version of F can be defined as tF with partials at $tf, 2tf, ..., ntf$ and amplitudes $v_1, v_2, ..., v_n$. The roughness $D_F(t)$ generated by the spectrum F is defined in function (3) and the shape of this function is shown in Figure 2. [4] This figure shows the comparison from a base tone to its seven overtones.

$$D_F(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} d(if, tjf, v_i, v_j), \quad (3)$$

3. DISSONANCE INDEX OF TRIADS

Thus far, we have introduced the preceding works concerning the dissonance value between the intervals. In this section, we propose our new definition of the dissonance value for triads. Given three tones with the ratio of intervals $1 < t_1 < t_2$, we add up the three values of (3) as function (4) and draw Figure 3 based on this function.

[4] The figure is a reproduction, appearing in [4].

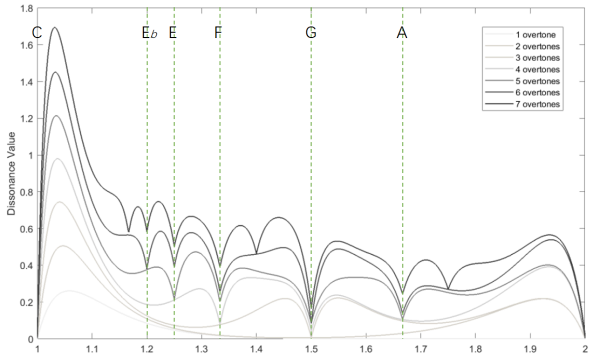

Figure 2: Dissonance Value for Intervals, Dependent on Base Frequency

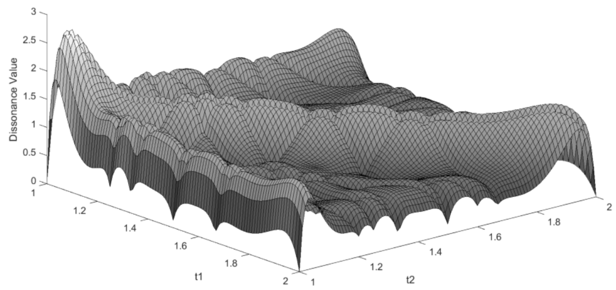

Figure 3: 3D Representation of Interval Dissonance in Triads

$$D_F(t_1, t_2) = D_F(t_1) + D_F(t_2) + D_{t_1 F}(\frac{t_2}{t_1}) \quad (4)$$

where

$$D_F(t_1) = \sum_{i=1}^{n} \sum_{j=1}^{n} d(if, jt_1 f, v_i, v_j),$$

$$D_F(t_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} d(if, jt_2 f, v_i, v_j),$$

$$D_{t_1 F}(\frac{t_2}{t_1}) = \sum_{i=1}^{n} \sum_{j=1}^{n} d(it_1 f, \frac{t_2}{t_1} jf, v_i, v_j).$$

We have employed twelve major keys and twelve minor keys, each of which includes seven triads on diatonic notes, including three major triads, three minor triads, and one diminished triad (vii°). In this paper, we have omitted the harmonic and melodic minor scales. Therefore, since a pair of parallel keys consists of the same set of chords, we take 12 group of chords as research objects to evaluate the music temperaments.

In the first attempt, the average value of 12 group of chords in each temperament are calculated with our dissonance value model. According to the ratios in Table 1, we consider the frequencies of an octave starting from the central C, for three typical temperaments (Pythagorean tuning, just intonation and equal temperament) as examples. The results are shown in Figure 4.

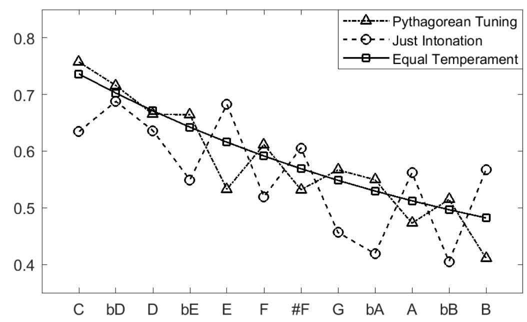

Figure 4: Relative Dissonance of Chords in a Major Scale, Compared to Equal Temperament

As we can see, even the equal temperament does not guarantee consistent values in different keys. The reason for this is obviously due to the different frequencies of the base pitches, resulting in different critical bandwidths. Therefore, in this paper, we choose to preserve the frequency ratios and to transpose the base pitch into a certain fixed value F_0. The adjusted model should no longer be called the dissonance level but an index to compare the consonance degree among different triads, and thus we call it the Dissonance Index of Triads (DIT) (5), hereafter.

$$DIT(t_1, t_2) = D_{F_0}(t_1, t_2) \tag{5}$$

We set the frequency of F_0 to 263Hz, that is an approximate frequency of the central C.

4. DIT IN TEMPERAMENTS

This chapter expounds the results of DIT values of triads in different temperaments. According to Figure 2, this research employs overtones upon the seven diatonic tones. We fix the ratio of amplitudes to be a geometric progression with 0.9 as mentioned before.

4.1 The Weighted Chords in Each Key

Prior to the evaluation, we have given the following weights $[0.86 : 0.26 : 0.17 : 0.73 : 0.73 : 0.56 : 0]$ on each triads on the diatonic scale, based on the usage of the chords in 1300 popular songs [5] after transposed to C-major. Note that our objective here is to compare the average consonant level of various chords in different keys, and not to assess the human feeling; thus the transposition is a kind of approximation. The average consonant level is shown in Table 2, for each key and temperament.

For more intuitive and clear understanding, we compare the DIT between two triads chosen from the two different temperaments. In Figure 5, the horizontal axis of each graph represents the keys while the vertical axis represents the DIT value. Since the DIT value is adjusted from the dissonance value, the lower the DIT value is the more consonant the key is.

The comparison of the Sanfen Sunyi-fa and Pythagorean tuning is shown in the upper-left graph in Figure 5. We can see that because they share the same process of generation except for the location of the wolf fifth, the difference of DIT shifts in parallel. Similarly, the quarter-comma mean-tone and the conventional quarter-comma mean-tone, shown in the upper-right graph has the same property.

In the lower-left graph in Figure 5, we can hardly find the big difference between the two well temperaments along the horizontal trends. That is to say, the well temperaments tend to change slightly between adjacent keys, trying to distribute the dissonance reasonably to keep the balance, and thus there are no peaks in dissonance. They also especially ensures some commonly used keys such as C-major, D-major, F-major, and G-major, are in better consonance.

The lower-right graph in Figure 5shows the comparison between the just intonation and the equal temperament. We can see that the equal temperament presents a perfect horizontal line, which proves that it will sound always the same no matter what key it is. But, its DIT value is also relatively higher with no keys in better consonance. The just intonation has the lowest DIT value of all the results in a few keys such as C and A♭, but the line goes up and down steeply and the dissonant keys are also obvious. We can easily read in this figure that the equal temperament and the just intonation are the two extremes in modulation and consonance.

4.2 Consonance or Modulation

In accordance with the position of the wolf fifth or other adjustments, there are also difference in what keys they prefer. In fact, there is a difference in the degree of commonality of each key, which means we had better take the weights of keys into consideration. A survey of "The Most Popular Keys of All Music" [6] on Spotify™ in 2005 showed the data in Table 3. Here, we put a major key and its parallel key together because they share a common set of the diatonic chords.

Taking both the weights of keys and the diatonic chords into consideration, we visualize the balance between the consonance and modulation of temperaments as in Table 4, which is plotted in Figure 6. The horizontal axis shows the average DIT value with weights, which refers to the consonance level of the temperament, while the vertical axis represents the average standard deviation of chords and represents the smoothness of modulation. The lower the value is, the more easily the temperament can modulate. It is obvious that just intonation is outstanding at consonance but worse in modulation, and equal temperament is the opposite, that is, the easiest in modulation but the worst in consonance. Pythagorean tuning and Sanfen Sunyi-fa are staying at a similar level on modulation, and there are slight difference because of the weights in keys.

At last, the well-temperaments obtained a very good result; Kirnberger temperament does the best in consonance than all the other temperaments except for just intonation, and Werckmeister wins on modulation. Note that they locate just near on the line linked by just intonation and

[5] https://www.hooktheory.com/theorytab/

[6] https://insights.spotify.com/us/2015/05/06/most-popular-keys-on-spotify/

	SS	PT	JI	ET	QM	CQM	$Wm.$	$Kb.$
C	0.9407	0.9478	0.8366	0.9276	0.8598	0.8598	0.8861	0.8668
G	0.9478	0.9478	0.8514	0.9276	0.8598	0.8598	0.9084	0.8463
D	0.9478	0.9250	0.9008	0.9276	0.9120	0.8598	0.9251	0.8744
A	0.9478	0.8969	0.9811	0.9276	0.9926	0.8598	0.9261	0.9161
E	0.9478	0.8521	1.0003	0.9276	1.0365	0.9120	0.9325	0.9443
B	0.9478	0.8678	1.0151	0.9276	1.0544	0.9926	0.9385	0.9417
$F\sharp$	0.9478	0.8956	0.9735	0.9276	1.0055	1.0365	0.9487	0.9435
$C\sharp$	0.9250	0.9407	0.9215	0.9276	0.9382	1.0544	0.9478	0.9453
$G\sharp$	0.8969	0.9478	0.8366	0.9276	0.8598	1.0055	0.9433	0.9485
$D\sharp$	0.8521	0.9478	0.8514	0.9276	0.8598	0.9382	0.9321	0.9478
$A\sharp$	0.8678	0.9478	0.8560	0.9276	0.8598	0.8598	0.9051	0.9380
F	0.8956	0.9478	0.8720	0.9276	0.8598	0.8598	0.8865	0.9096

Table 2: Average Consonant Level of Chords in Different Keys in Each Temperament

Mjor Keys		Parallel Keys		Total
C	10.20%	a	4.80%	15.00%
G	10.70%	e	4.20%	14.90%
D	8.70%	b	4.20%	12.90%
A	6.10%	$f\sharp$	2.50%	8.60%
E	3.60%	$c\sharp$	2.10%	5.70%
B	2.60%	$g\sharp$	1.20%	3.80%
$F\sharp$	2.70%	$d\sharp$	0.90%	3.60%
$C\sharp$	6.00%	$a\sharp$	3.20%	9.20%
$G\sharp$	4.30%	f	3.00%	7.30%
$D\sharp$	2.40%	c	2.40%	4.80%
$A\sharp$	3.50%	g	2.60%	6.10%
F	5.30%	d	2.60%	7.90%

Table 3: Usage of Keys in Popular Music

	Avg	SD
Sanfen Sunyi-fa	0.9245	0.0446
Pythagorean Tuning	0.9277	0.0425
Just Intonation	0.8981	0.0868
Equal Temperament	0.9276	0.0000
Quarter-Comma Meantone	0.9154	0.0917
Conventional QC Meantone	0.9267	0.0972
Werckmeister	0.9227	0.0284
Kirnberger	0.9120	0.0491

Table 4: DIT Results (SD is the standard deviation)

equal temperament, which implies that they are balanced between the two criteria.

Here, we have to note that this graph is biased by the usage of chords found in Spotify™ database, *i.e.*, the usage of chords are more inclined to that in the modern popular music. On the contrary, the mean-tone, dotted on the upper-right corner in the figure, was invented to obtain the clear resonance of the major third preferred in classicist age. It is easily guessed that if we employ the database of classical music the tendency would be different. The variety of distribution of dots in this space would surely reflect the difference of music genre, and this is our future work.

5. DISCUSSION AND CONCLUSION

We have proposed an index to show the numerical consonance level of triad, DIT, and have compared the difference of the level in various temperaments. Since chords on a scale may have different significance, we have weighted them by the number of appearance. The resultant difference has been visualized in various graphs.

Nowadays, we do not need to stick to the five-line staff based on 12-tone equal temperament in composing music since actual performance should tolerate micro-tones, out-of-tune tones, portament, vibrating tunes, and so on. This tendency would be more salient in computer music age in future. It may be high time for us to reconsider traditional temperaments, to give special savors in music or to escape temporarily from the equal temperament, so that we should know the concrete difference in temperaments.

Acknowledgments

This work is supported by JSPS kaken 16H01744.

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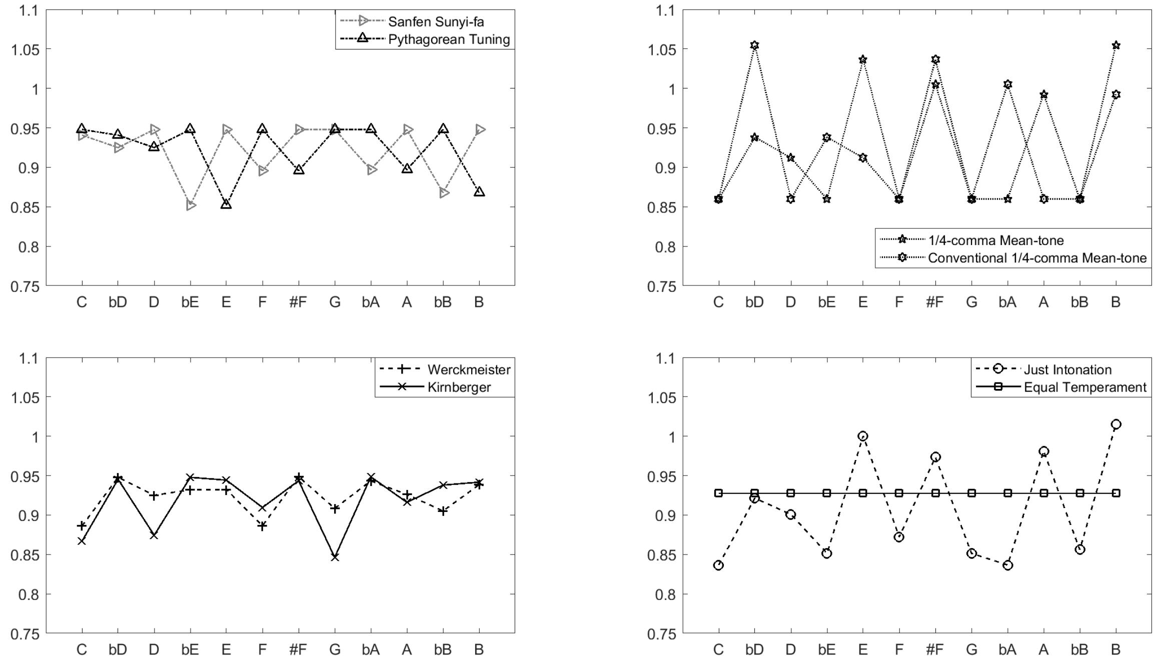

Figure 5: Comparison of Temperaments in Keys versus DIT Values; (upper-left) Sanfen Sunyi-fa vs Pythagorean, (upper-right) two mean-tones, (lower-left) two well temepraments, and (lower-right) the just intonation and the equal temperament

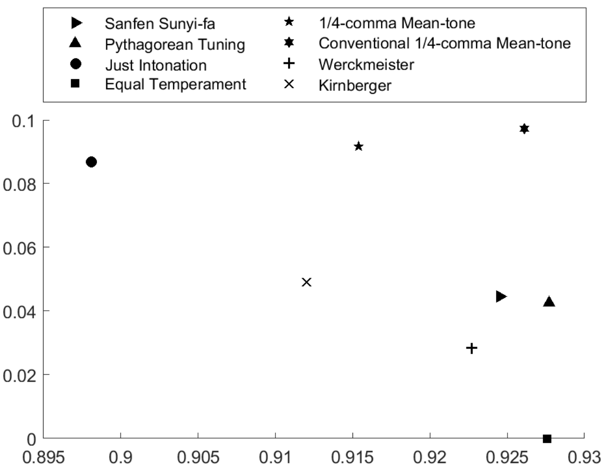

Figure 6: Consonance and Modulation Distribution

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