Representations of self-coupled modal oscillators with time-varying frequency

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ABSTRACT

In this work we examine a simple mass-spring system in which the natural frequency is modulated by its own oscillations, a self-coupling that creates a feedback system in which the output signal "loops back" with an applied coefficient to modulate the frequency. This system is first represented as a mass-spring system, then as an extension of well-known frequency modulation synthesis (FM) coined "loopback FM", and finally, as a closed-form representation that has a form similar to the transfer function of a "stretched" allpass filter with time-varying delay, but with the fundamental difference that it is used here as a time-domain signal, the real part of which is the sounding waveform. This final representation allows for integration of instantaneous frequency in the FM representation and ultimately a mapping from its parameters to those of loopback FM. In addition to predicting the sounding frequency (pitch glides) of loopback FM for a given carrier frequency and time-varying loopback coefficient, or equivalently of the self-coupled oscillator for a given natural frequency and coupling coefficient, the closed form representation is seen to be a more accurate representation of the system as it does not introduce a unit-sample delay in the feedback loop, nor is it as numerically sensitive to sampling rate.

1. INTRODUCTION

It is well known that introducing nonlinearities into a linear system may contribute computational complexity making it prohibitive for real-time use [1,2]. If, however, the aim is to apply the dynamic sound effects of nonlinear coupling to a synthesized sound, prioritizing real-time parametric control over acoustic accuracy, there are representations in the literary canon of parametric abstract synthesis techniques that can be explored. In spirit and purpose similar to [3], this work explores the relationship between a physically self-coupled oscillator to the well-known abstract synthesis technique, frequency modulation (FM).

As shown in Section 2, a simple physical model of a two-degree-of-freedom (2-DOF) mass-spring system that exhibits modal coupling behaviour, can be "abstracted" and represented as a simplified self-coupled oscillator, one in which the mass influences its own oscillation in a feedback system. Discretization of this oscillator's displacement, yielding a second-order bandpass filter, is problematic when the frequency, and thus the filter coefficient dependent on frequency, is made time varying. The system is shown in Section 4 to be strongly related to FM, and to resolve issues of filter instability, the self-coupled oscillator is presented in terms of a variant of FM so called "loopback FM" [4] to distinguish it from the related, but distinct, "feedback FM" [5]. Finally, the closed-form representation of the loopback FM oscillator is given, offering improved numerical accuracy, eliminating the need for a unit-delay in the feedback loop and, perhaps most advantageous for musical applications, revealing the nonlinear oscillator's sounding frequency. The musical application of this work is explored in a related paper by the same authors, whereby a modal synthesis model of percussion instruments is implemented using loopback FM oscillators, allowing for a linear model to be enhanced by the rich and dynamic sounds caused by nonlinear modal coupling that are characteristic of these instruments [6].

2. A COUPLED OSCILLATOR

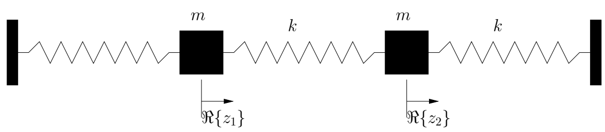

Figure 1. A two-degree-of-freedom oscillator having two masses and three springs of equal value, where the displacement of each mass is given by $\Re\{z_{1,2}\}$.

The equations of motion for a two-degree-of-freedom mass-spring oscillator with mass m and spring constant k are

$$\begin{aligned} m\ddot{z}_1 + kz_1 + k(z_1 - z_2) &= 0 \\ m\ddot{z}_2 + kz_2 + k(z_2 - z_1) &= 0, \end{aligned} \tag{1}$$

with displacement of each mass being given by $\Re\{z_{1,2}(t)\}$, and $z_{1,2}(t)$ having assumed solutions

$$z = Ae^{j\omega t}, \dot{z} = j\omega Ae^{j\omega t}, \ddot{z} = -\omega^2 Ae^{j\omega t}, \tag{2}$$

and where (1) has 2 natural modes of oscillations: one where $A_1 = A_2$:

$$-\omega^2 A_1 + \omega_0^2 A_1 + \omega_0^2 \cancelto{0}{(A_1 - A_2)} = 0 \quad \text{and} \quad \omega = \omega_0 \tag{3}$$

and the other where $A_1 = -A_2$:

$$-\omega^2 A_1 + \omega_0^2 A_1 + \omega_0^2 \cancelto{2A_1}{(A_1 - A_2)} = 0 \quad \text{and} \quad \omega = \sqrt{3\omega_0^2}, \tag{4}$$

where $\omega_0 = \sqrt{k/m}$. In addition to the 2 natural modes, $z_{1,2}$ may exhibit coupled behaviour (e.g. given a specific set of initial conditions) in which one mass influences the oscillations of the other. To explore coupled behaviour, we begin with a generalized parametric expression in which frequency is modulated by the oscillations of the system

$$\omega = \omega_0 + d_1\Re\{z_1\} + d_2\Im\{z_1\} + d_3\Re\{z_2\} + d_4\Im\{z_2\}, \quad (5)$$

where coupling coefficients $d_{1,2,3,4}$ specify the amount of frequency deviation contributed by the oscillations of each mass (considering both real and imaginary parts of complex $z_{1,2}$). In this work, a special simplified case of (5) is explored where $d_{2,3,4} = 0$ and $\omega = \omega_0 + d_1\Re\{z_1\}$, and the oscillator is merely coupled to itself, creating a system that will be later referred to in Section 4 (and was previously coined in [4]) as “loopback FM”.

Since frequency ω is now made time varying, the relationship between instantaneous frequency $\omega_i(t)$ and instantaneous phase $\theta_i(t)$,

$$\omega_i(t) = \frac{d}{dt}\theta_i(t) \quad \text{and} \quad \theta_i(t) = \int_0^t \omega_i(t)\, dt, \quad (6)$$

must be considered before defining assumed solution z_1:

$$z_1(t) = \exp\left(j\int_0^t (\omega_0 + d_1\Re\{z_1(t)\})\, dt\right) \quad (7)$$

a system for which sounding frequency is not as easily predicted as in (2). Furthermore, if d_1 is made time varying, the sounding frequency will change over time, resulting in a pitch glide—a known characteristic of nonlinearly coupled systems—having a trajectory dependent on the nature of the function $d_1(t)$.

2.1 Assumed solution

Given the more general assumed solution $z_1(t) = e^{j\theta(t)}$, its first and second derivatives with respect to time are given by

$$\begin{aligned}
\dot{z}_1(t) &= j\dot{\theta}(t)z_1(t), && (8)\\
\ddot{z}_1(t) &= j\ddot{\theta}(t)z_1(t) + j\dot{\theta}(t)\dot{z}_1(t) \\
&= (j\ddot{\theta}(t) - \dot{\theta}(t)^2)z_1(t), && (9)
\end{aligned}$$

with the angle and its derivatives given by

$$\begin{aligned}
\theta(t) &= \int_0^t \omega_0 + d_1\Re\{z_1(t)\}\, dt, && (10)\\
\dot{\theta}(t) &= \omega_0 + d_1\Re\{z_1(t)\}, && (11)\\
\ddot{\theta}(t) &= d_1\Re\{\dot{z}_1(t)\} \\
&= d_1\Re\{j\dot{\theta}(t)z_1(t)\} \\
&= d_1\Re\{j\dot{\theta}(t)\Re\{z_1(t)\} - \dot{\theta}(t)\Im\{z_1(t)\}\} \\
&= -d_1\dot{\theta}(t)\Im\{z_1(t)\}. && (12)
\end{aligned}$$

The equation of motion adapted from (1) for a single mass-spring oscillator

$$m\ddot{z}_1(t) + kz_1(t) = 0 \quad (13)$$

in which the spring constant is modulated such that

$$\begin{aligned}
\sqrt{k(t)/m} &= \omega_0 + d_1z_1(t) && (14)\\
k(t) &= m(\omega_0 + d_1z_1(t))^2 && (15)
\end{aligned}$$

may be represented as

$$\ddot{z}_1(t) + (\omega_0 + d_1z_1(t))^2z_1(t) = 0, \quad (16)$$

which, having additional terms $2\omega_0d_1z_1^2$ and $d_1^2z_1^3$, is now nonlinear in z_1. To verify that the interpretation of $k(t)$ given in (14-15) satisfies the equation of motion for a self-coupled (feedback) oscillator, equation (9) is first substituted for $\ddot{z}_1(t)$ in (16),

$$j\ddot{\theta}(t) - \dot{\theta}(t)^2 = -(\omega_0 + d_1z_1(t))^2, \quad (17)$$

and (12) substituted for $\ddot{\theta}(t)$ in (17) to yield

$$\begin{aligned}
-jd_1\dot{\theta}(t)\Im\{z_1(t)\} - \dot{\theta}(t)^2 &= -(\omega_0 + d_1z_1(t))^2 \\
\dot{\theta}(t)\left(jd_1\Im\{z_1(t)\} + \dot{\theta}(t)\right) &= (\omega_0 + d_1z_1(t))^2 && (18)
\end{aligned}$$

where, by (11), the LHS parenthetical expression in (18) is

$$jd_1\Im\{z_1(t)\} + \omega_0 + d_1\Re\{z_1(t)\} = \omega_0 + d_1z_1(t) \quad (19)$$

to finally yield

$$\begin{aligned}
\dot{\theta}(t)(\omega_0 + d_1z_1(t)) &= (\omega_0 + d_1z_1(t))^2 \\
\dot{\theta}(t) &= \Re\{\omega_0 + d_1z_1(t)\}, && (20)
\end{aligned}$$

showing an instantaneous frequency equal to (11) when it is assumed to be real, thus further showing $\sqrt{k(t)/m} = \omega_0 + d_1z_1(t)$ satisfies the equation of motion.

2.2 Implementation of mass-spring oscillator

One known solution for the discretization of the mass-spring oscillator is using the trapezoidal rule for numerical integration (or bilinear transform). A version of (13) that is driven by force function $F_k(t) = F(t)/k$:

$$\ddot{z}_1(t) + \omega_0^2z_1(t) = F_k(t), \quad (21)$$

has s-transform

$$s^2Z_1(s) + \omega_0^2Z_1(s) = F_k(s), \quad (22)$$

and transfer function

$$H(s) = \frac{Z_1(s)}{F_k(s)} = \frac{1}{s^2 + \omega_0^2}, \quad (23)$$

which, when taking the z-transform by substituting s with $c\frac{1 - z^{-1}}{1 + z^{-1}}$, where $c = 2/T$ without prewarping, yields

$$H(z) = \left(\frac{1}{c^2 + \omega_0^2}\right)\frac{1 + 2z^{-1} + z^{-2}}{1 - 2\frac{c^2 - \omega_0^2}{c^2 + \omega_0^2}z^{-1} + z^{-2}}. \quad (24)$$

The numerator and denominator polynomials of (24) may be expressed in polar form as,

$$\begin{aligned}
B(z) &= \left(1 + z^{-1}\right)^2 && (25)\\
A(z) &= \left(1 - az^{-1}\right)\left(1 - a^*z^{-1}\right), && (26)
\end{aligned}$$

where roots of $A(z)$ (poles of $H(z)$) are the complex conjugate pair having sum

$$2\frac{c^2-\omega_0^2}{c^2+\omega_0^2} = \frac{(c-j\omega_0)^2+(c+j\omega_0)^2}{(c+j\omega_0)(c-j\omega_0)} = \underbrace{\frac{c-j\omega_0}{c+j\omega_0}}_{a} + \underbrace{\frac{c+j\omega_0}{c-j\omega_0}}_{a^*}. \tag{27}$$

The gain of filter $H(z)$ is given by

$$G(\omega) = |H(\omega)| = \frac{1}{c^2+\omega_0^2}\frac{|B(\omega)|}{|A(\omega)|}, \tag{28}$$

where

$$\begin{aligned} |B(\omega)| &= \left|\left(e^{-j\omega T/2}\left(e^{j\omega T/2}+e^{-j\omega T/2}\right)\right)^2\right| \\ &= \left|e^{-j\omega T}\left(2+e^{j\omega T}+e^{-j\omega T}\right)\right| \\ &= 2(1+\cos(\omega T)), \qquad (29) \\ |A(\omega)| &= \left|e^{-j\omega T/2}\left(e^{j\omega T/2}-ae^{-j\omega T/2}\right)\right| \times \\ &\qquad \left|e^{-j\omega T/2}\left(e^{j\omega T/2}-a^*e^{-j\omega T/2}\right)\right| \\ &= \left|e^{-j\omega T}\left(-a-a^*+e^{j\omega T}+e^{-j\omega T}\right)\right| \\ &= \left|-2\left(\frac{c^2-\omega_0^2}{c^2+\omega_0^2}-\cos(\omega T)\right)\right|, \qquad (30) \end{aligned}$$

and (28) reduces to

$$G(\omega) = \frac{1+\cos(\omega T)}{|c^2-\omega_0^2-(c^2+\omega_0^2)\cos(\omega T)|}. \tag{31}$$

Transfer function (24) is a linear-time-invariant second-order bandpass filter having, as shown by (31), a spectral peak in the magnitude at ω_0 and taking the inverse transform of (24) yields an undamped sinusoidal oscillator that closely matches (21). Though (24) is well behaved for static ω_0, it has problems when made time varying. It could be made tuneable by introducing a loss as in [7], but this would have consequences when placed in a feedback system where the loss would accrue.

2.3 Discrete-Time Complex Oscillator

A point in the complex plane $z_s(0) = Ae^{j\phi_0}$ can be made to rotate with angle $\omega_i T$ via a complex multiply

$$z_s(1) = e^{j\omega_i T}Ae^{j\phi_0}, \tag{32}$$

or equivalently, as shown in [4], using a power preserving rotational matrix. If ω_i is static (indicated here by subscript s), regular rotations every time sample $n = 0, 1, ..., N-1$ produces an oscillator given by the complex sinusoid:

$$z_s(n) = (e^{j\omega_i T})^n Ae^{j\phi_0} = Ae^{j(\omega_i nT+\phi_0)}, \tag{33}$$

having instantaneous phase $\omega_i nT + \phi_0$, initial phase ϕ_0, and instantaneous angular frequency ω_i. If however, ω_i is made time varying, the representation in (33) no longer applies and the relationship between frequency and phase given in (6) must be considered before defining the oscillator. For example, if the oscillator frequency changes linearly from ω_1 to ω_2 over T_d seconds, the instantaneous phase would be given by

$$\int_0^t \left(\frac{\omega_2-\omega_1}{T_d}t+\omega_1\right)dt = \frac{\omega_2-\omega_1}{2T_d}t^2+\omega_1 t+C, \tag{34}$$

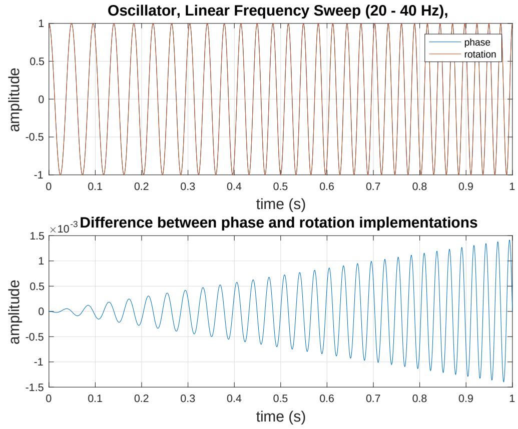

Figure 2. Though implementing an oscillator via a rotation of its previous sample (akin to a numerical integration of frequency) produces a similar result to implementing the phase analytically (top), there is a difference, and the error observed (bottom) can compound.

yielding the discrete-time oscillator (by substitution $t \to nT$) notably different from (33):

$$z_l(n) = \exp\left[j\left(\left(\frac{\omega_2-\omega_1}{2T_d}nT+\omega_1\right)nT+\phi_0\right)\right], \tag{35}$$

where initial phase ϕ_0 is set to the constant of integration C. It is possible to implement this oscillator via sample-by-sample rotations of angle $\omega_i(n)T$ though *not* of an initial complex value as in (33), but rather of its current state:

$$\begin{aligned} e^{j\omega_i(n)T}z_l(n-1) &= e^{j\omega_i(n)T}e^{j\omega_i(n-1)T}, ..., e^{j\omega_i(0)T}Ae^{j\phi_0} \\ &= \exp\left[j\sum_{m=0}^{n}\omega_i(m)T\right]Ae^{j\phi_0}. \qquad (36) \end{aligned}$$

The summation in (36) can also be viewed as a numerical integration of the instantaneous frequency, which, as shown in Figure 2, comes at a cost of numerical error when compared to using the instantaneous phase (34) directly. Though in many cases this error is negligible (and inaudible), it is the reason why, as discussed in the following section, it is often preferable to implement frequency modulation (FM) as phase modulation (PM), especially in cases where networks of multiple carriers can cause such error to compound.

3. FM/PM REPRESENTATION

In the well-known synthesis technique first introduced by Chowning [8], the frequency/phase of an oscillator may be made to change sinusoidally, introducing sidebands about a carrier frequency and changing the sound's spectrum in a way that's dependent on the amplitude, phase, and frequency of the modulating sinusoid. In frequency modulation (FM) synthesis, a carrier oscillator has a center frequency ω_c that is modulated by a sinusoid having amplitude d and frequency ω_m, yielding instantaneous frequency

$$\omega_i(t) = \omega_c + d\cos(\omega_m t), \tag{37}$$

where d determines the oscillator's peak frequency deviation from ω_c. Notably, (37) has a form very similar to (5) with $d_{2,3,4} = 0$, and this will be developed in the next section. The corresponding instantaneous phase is obtained by integrating (37) according to (6), yielding

$$\theta_i(t) = \int_0^t \omega_i(t)\, dt = \omega_c t + \frac{d}{\omega_m}\sin(\omega_m t) + \phi_c, \quad (38)$$

showing that FM may be equivalently expressed as phase modulation (PM), where it is the initial phase term that is sinusoidally time varying,

$$\phi(t) = I\sin(\omega_m t) + \phi_c, \quad (39)$$

with amplitude

$$I = \frac{d}{\omega_m}, \quad (40)$$

a value known as the *index of modulation* because of how it influences the magnitude of sidebands at $f_c \pm kf_m$ in the resulting spectrum. FM synthesis is frequently implemented as PM, preferred because of improved numerical properties (such as those illustrated in Figure 2) and accuracy less dependent on sampling rate, with the real signal being given by

$$x_c(t) = \cos(\omega_c t + I\sin(\omega_m t)), \quad (41)$$

or, as the real part of the complex exponential sinusoids,

$$x_c(t) = \Re\{z_c(t)\} = \Re\left\{e^{j(\omega_c t + \Im\{z_m(t)\})}\right\}, \quad (42)$$

where

$$\Im\{z_m(t)\} = I\sin(\omega_m t). \quad (43)$$

Using the complex form has the power-preserving advantage discussed above and in [4], and allows FM to be represented as a sample-by-sample rotation of its current state, shown by beginning with (42) at time sample $n-1$:

$$z_c(n-1) = e^{j(\omega_c (n-1)T + \Im\{z_m(n-1)\})}, \quad (44)$$

then adding and subtracting $\Im\{z_m(n)\}$ to its argument

$$\begin{aligned}\angle z_c(n-1) &= j(\omega_c nT + \Im\{z_m(n)\} - \omega_c T \\ &\quad -\Im\{z_m(n)\} + \Im\{z_m(n-1)\}) \\ &= \angle z_c(n) \\ &\quad -j(\omega_c T + \Im\{z_m(n) - z_m(n-1)\})\end{aligned} \quad (45)$$

so that $z_c(n-1)$ may be represented first as a multiplication by $z_c(n)$

$$z_c(n-1) = z_c(n)e^{-j(\omega_c T + \Im\{z_m(n) - z_m(n-1)\})}, \quad (46)$$

and then finally in its causal form

$$z_c(n) = e^{j(\omega_c T + \Im\{z_m(n) - z_m(n-1)\})} z_c(n-1). \quad (47)$$

Notably, this result is equivalent to taking the derivative of the phase with respect to continusous-time t to produce instantaneous frequency,

$$\omega_i(t) = \frac{d}{dt}\left(\omega_c t + \Im\{z_m(t)\}\right) = \omega_c + \frac{d}{dt}\Im\{z_m(t)\}, \quad (48)$$

then using a finite different approximation to obtain its discrete-time form

$$\omega_i(n) = \omega_c + \frac{\Im\{z_m(n)\} - \Im\{z_m(n-1)\}}{T}, \quad (49)$$

which, when normalized by the sampling period T yields the angle of rotation in (47).

4. SELF COUPLING AND LOOPBACK FM

Applying the above to the self-coupled oscillator in (5) where $d_{2,3,4} = 0$ and the instantaneous frequency is $\omega_i(t) = \omega_0 + d_1\Re\{z_1(t)\}$, it is evident from (44 - 47) that this system may be expressed as a sample-by-sample rotation of its current state, where the carrier oscillator is "looped back" to serve as the modulator of its frequency, with added unit sample delay necessary for implementation:

$$z_c(n) = e^{j\,(\omega_c + B\omega_c\Re\{z_c(n-1)\})\,T} z_c(n-1), \quad (50)$$

and modulation amplitude $B\omega_c = d$ determines the peak frequency deviation from ω_c, while the loopback coefficient B functions as the index of modulation according to (40).

A more accurate representation of (50) is expected of one in which a delay of one sample is not required and which does not essentially implement a numerical approximation of the system's instantaneous phase, as shown by (47 - 49). Integrating the instantaneous frequency $\omega_c + B\omega_c\Re\{z_c(t)\}$ with respect to continuous-time t yields an alternate representation of the system in which the corresponding instantaneous phase is given by

$$\begin{aligned}\theta_i(t) &= \int_0^t \omega_c + B\omega_c\Re\{z_c(t)\}dt \\ &= \omega_c t + B\omega_c\Re\left\{\int_0^t z_c(t)\, dt\right\}.\end{aligned} \quad (51)$$

Though the integral term in (51) may be implemented via numerical integration to yield the discrete-time representation of instantaneous phase:

$$\theta_i(n) = \omega_c nT + B\omega_c T\Re\left\{\sum_{k=0}^{n-1} z_c(k)\right\}, \quad (52)$$

this solution does not improve upon—and in fact is exactly equal to—equation (50) when incorporated into the phase modulation representation $e^{j\theta_i(n)}$. Furthermore, it does not provide greater understanding of the system's behaviour, and in particular, reveal at what frequency it will sound. It is preferable, therefore, to solve (51) analytically.

4.1 Analytic Solution to $\theta_i(t)$ for static B

Figure 3 plots the real part of $z_c(n)$ given in (50), showing a periodic signal having a period of M samples and a sounding frequency of $f_0 = f_s/M$ Hz, a signal that can also be described by the real part of

$$z_0(n) = \frac{b_0 + e^{j\omega_0 nT}}{1 + b_0 e^{j\omega_0 nT}}, \quad (53)$$

where $\omega_0 = 2\pi f_s/M$. Equation (53) is similar in form to the transfer function of the "stretched" allpass filter used in

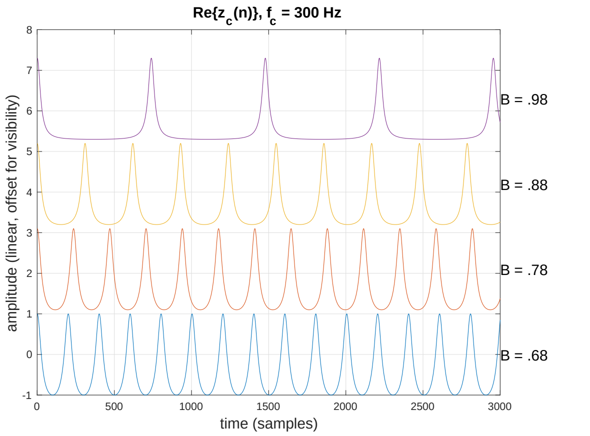

Figure 3. The real part of $z_c(n)$ given in (50) is plotted (with offset) for 4 linearly spaced values of B, showing a *nonlinear* relationship to resulting period in samples M (and sounding frequency $f_0 = f_s/M$ Hz,$f_s = 44.1$ kHz).

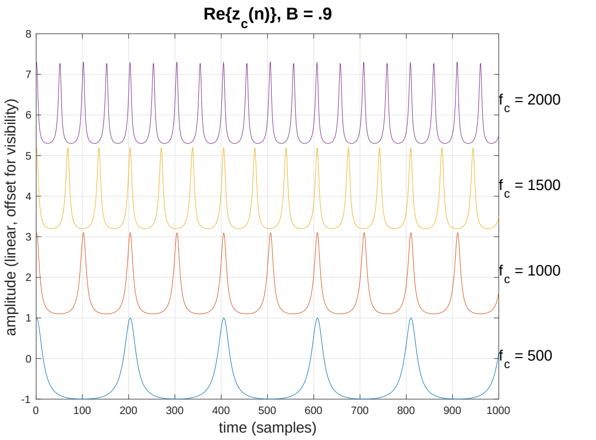

Figure 4. The real part of $z_c(n)$ given in (50) is plotted with 4 linearly spaced values of $f_c = \omega_c/(2\pi)$ showing a *linear* relationship with resulting period in samples M (and sounding frequency $f_0 = f_s/M$ Hz, $f_s = 44.1$ kHz).

[9], though here it is used as a time-domain signal that is a function of time sample n—a complex oscillator of which we ultimately take the real part to produce the sounding waveform. It is also interesting to observe a similarity between the waveforms in Figures 3-5 and those produced by feedback amplitude modulation (FBAM) [10] for input $\cos(\omega_0 nT)$, as well as the related time series in [11]. Though the pulse shape and offset are indeed different, their similarity does suggest further study of their relationship would be worthwhile.

Though Figure 6 shows $z_0(n)$ and $z_c(n)$ diverging for increased values of $f_c = \omega_c/(2\pi)$ and B (not shown), this is improved with increased sampling rate (reducing numerical error as well as the effect of the unit-sample delay in (50)), providing confidence that $z_0(n)$ is actually the preferred and more accurate solution to the self-coupled oscillator. With this assumption, the integral of $z_c(t)$ with respect to continuous-time t in (51) may now be expressed analytically by the integral of continuous-time $z_0(t)$ (obtained by substitution $nT \rightarrow t$ in $z_0(n)$ given in (53)) which, as shown in Appendix A for static ω_0 and b_0, is

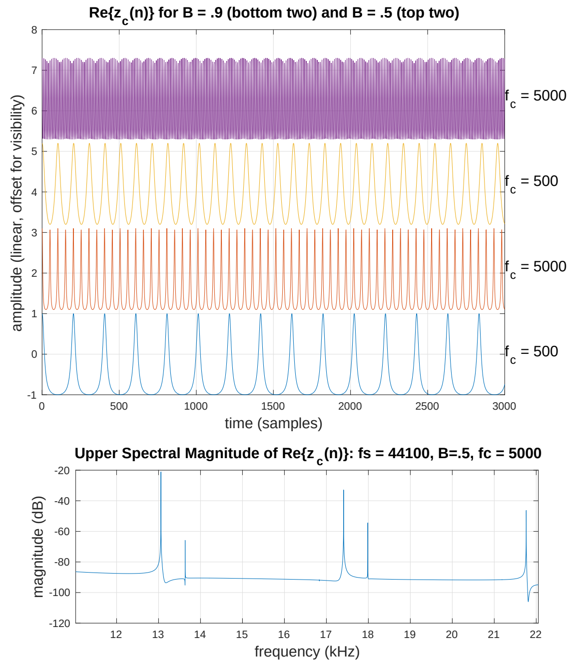

Figure 5. Higher values of f_c with lower values of B can produce signals that show low-frequency amplitude modulation (beating) most prominent here when $B = .5$ and $f_c = 5000$ Hz (top). The beat period ($\approx$ 80 samples) and beat frequency ($\approx f_s/80 = 551$, $f_s = 44.1$ kHz), is also visible as the frequency difference (bottom) between higher frequency components closer to the Nyquist limit $f_s/2$. This strongly suggests aliasing, and artifacts disappear when f_s is increased.

given by

$$\int_0^t z_0(t)\,dt = b_0 t + \frac{1-b_0^2}{j\omega_0 b_0}\log\left(1 + b_0 e^{j\omega_0 t}\right). \quad (54)$$

Representing the term inside the logarithm in polar form

$$1 + b_0 e^{j\omega_0 t} = A(t)e^{\phi(t)}, \quad (55)$$

where

$$A(t) = \sqrt{1 + 2b_0\cos(\omega_0 t) + b_0^2} \quad (56)$$

and

$$\phi(t) = \tan^{-1}\left(\frac{b_0\sin(\omega_0 t)}{1 + b_0\cos(\omega_0 t)}\right), \quad (57)$$

the real part of (54) may be expressed as

$$\begin{aligned}\Re\left\{\int_0^t z_0(t)\,dt\right\} &= b_0 t + \\ &\quad \Re\left\{\frac{1-b_0^2}{j\omega_0 b_0}\left(\log(A(t)) + j\phi(t)\right)\right\} \\ &= b_0 t + \frac{1-b_0^2}{\omega_0 b_0}\phi(t), \end{aligned} \quad (58)$$

and the final expression for phase $\theta_i(t)$ in (51) becomes

$$\begin{aligned}\theta_i(t) &= \omega_c t + B\omega_c \Re\left\{\int_0^t z_c(t)\,dt\right\}. \\ &= \omega_c t(1 + Bb_0) + B\omega_c\frac{1-b_0^2}{\omega_0 b_0}\phi(t). \end{aligned} \quad (59)$$

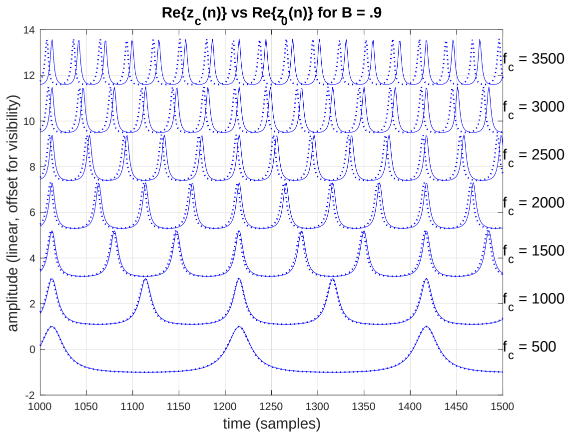

Figure 6. $\Re\{z_c(n)\}$ (solid) and $\Re\{z_0(n)\}$ (broken) show increasingly less agreement over time for higher f_c and higher B (not shown) as they drift out of phase.

for yet unknown values b_0 and ω_0, solved in terms of loopback FM parameters f_c and B in the following section.

4.2 Mapping b_0 and ω_0 to Loopback FM Parameters

Expressions for parameters b_0 and ω_0 may be obtained by first setting the angle of $z_0(t)$, well known but derived in Appendix B as

$$\angle z_0(t) = \omega_0 t - 2\tan^{-1}\left(\frac{b_0 \sin(\omega_0 t)}{1 + b_0 \cos(\omega_0 t)}\right), \qquad (60)$$

equal to the instantaneous phase of the loopback FM representation given in (59):

$$\omega_c t(1 + Bb_0) + B\omega_c \frac{1 - b_0^2}{\omega_0 b_0}\phi(t) = \omega_0 t - 2\phi(t), \qquad (61)$$

where $\phi(t)$ is given in (57). Setting linear terms on the left- and right-hand side (LHS and RHS) of (61) to be equal, yields one expressions for ω_0:

$$\omega_0 = \omega_c(1 + Bb_0), \qquad (62)$$

while setting LHS and RHS oscillating terms to be equal yields a second expression for ω_0:

$$\omega_0 = \frac{\omega_c B(1 - b_0^2)}{-2b_0}. \qquad (63)$$

Setting (62) equal to (63) yields the quadratic equation

$$Bb_0^2 + 2b_0 + B = 0, \qquad (64)$$

where b_0 is given in terms of loopback FM parameter B:

$$b_0 = \frac{\pm\sqrt{1 - B^2} - 1}{B}. \qquad (65)$$

Finally, substituting (65) into (62) yields an expression for ω_0 as a function of loopback FM parameters B and ω_c:

$$\omega_0 = \omega_c(1 + B\frac{\pm\sqrt{1 - B^2} - 1}{B}) = \pm\omega_c\sqrt{1 - B^2}. \qquad (66)$$

4.3 Allowing for Time-Varying Sounding Frequency

The derivation in the previous section assumes a static loopback variable B which, by (66), also produces a static sounding frequency ω_0 and static b_0. To produce a change in sounding frequency over time, B must be made time varying and the expression for $z_0(t)$ made more generalized:

$$z_0(t) = \frac{b_0 + e^{j\theta_0(t)}}{1 + b_0 e^{j\theta_0(t)}}, \qquad (67)$$

where the argument of the exponential terms is the integral with respect to time of time-varying frequency $\omega_0(t)$:

$$\theta_0(t) = \int_0^t \omega_0(t) = \pm \int_0^t \omega_c \sqrt{1 - B(t)^2}\, dt. \qquad (68)$$

and $\omega_0(t)$ is adapted from (66) for time-varying $B(t)$. Clearly the expression resulting from (68) is dependent on the nature of function $B(t)$.

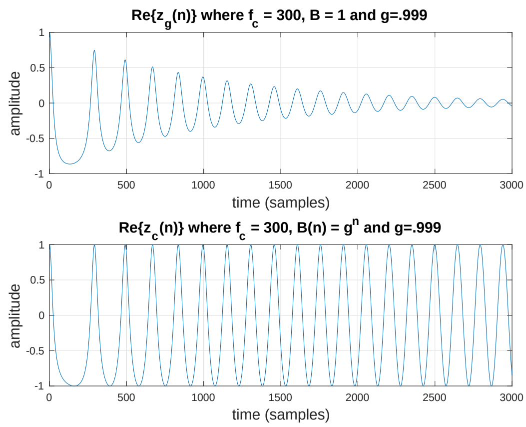

Figure 7. Waveform $\Re\{z_g(n)\}$ (top) show that introducing a scalar multiple g to the loopback FM equation (69) introduces a change in both sounding frequency and amplitude. Waveform $\Re\{z_c(n)\}$ (bottom) shows that setting the feedback coefficient to $B(n) = g^n$ produces the same pitch change but without imposing an amplitude envelope.

Consider the case where the complex oscillator is multiplied by a scalar value g such that when it is looped back, its amplitude envelope decays exponentially:

$$z_g(n) = ge^{j\,(\omega_c + \omega_c \Re\{z_g(n-1)\})\,T} z_g(n-1), \qquad (69)$$

where here $B = 1$. As shown in Figure 7, and evident from (69), the system will have both an amplitude envelope and a time-varying frequency, the latter equal to that of the loopback FM oscillator (50) in which B is made time varying with exponential function

$$B(n) = g^n, \quad n = 0, 1, ..., N-1. \qquad (70)$$

Representing loopback FM with time-varying $B(n)$ instead of (69) allows amplitude and (sounding) frequency envelopes to be divorced and independently described. If $B(n)$ is exponentially time varying according to (70), θ_0 may be adapted from (68) and represented as a function of discrete-time sample n:

$$\theta_0(n) = \int_0^n \omega_0(n) T\, dn = \pm \int_0^n \omega_c T \sqrt{1 - g^{2n}}\, dn, \qquad (71)$$

which, as shown in Appendix C, yields final expression

$$\theta_0(n) = \frac{\omega_c T}{\log(g)} \left(\sqrt{1-g^{2n}} - \tanh^{-1}(\sqrt{1-g^{2n}}) + C \right), \tag{72}$$

where C is an integration constant. Of course a different solution would result for $\theta_0(n)$ if $B(n)$ were made to change linearly with sample n:

$$B_l(n) = kn + l, \quad n = 0, 1, ...N, \tag{73}$$

yielding

$$\begin{aligned} \theta_0(n) &= \int_0^n \omega_c \sqrt{1 - B_l(n)^2}\, dn \\ &= \frac{\omega_c T}{2k} \left(B_l(n)\sqrt{1-B_l(n)^2} + \sin^{-1} B(n) \right) + C. \end{aligned} \tag{74}$$

Figure 8 shows the spectrum of $z_0(n)$ overlaid with a dark curve plotting time-varying fundamental frequency $f_0 = f_c\sqrt{1-B(n)}$ for $B(n) = g^n$ (top) and $B(n) = kn + l$ (bottom). The close fit between curve and lowest spectral harmonic shows that the fundamental frequency of loopback FM can be accurately predicted for both static and time-varying B and, in the latter case, use of (72) and (74) in expression for $z_0(n)$ in (67) is valid.

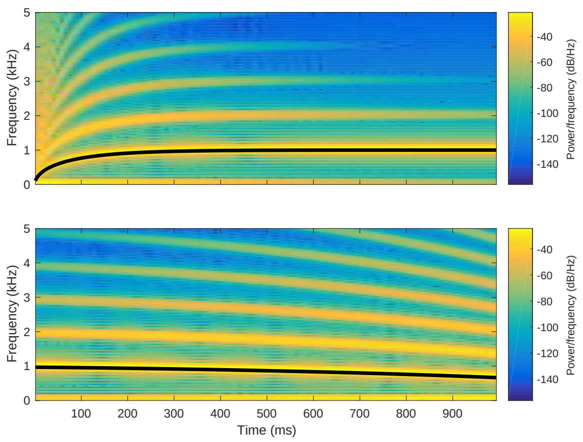

Figure 8. The spectrum of $z_0(n)$ with time-varying argument $\theta_0(n)$ overlaid with fundamental frequency $f_c\sqrt{1-B(n)}$ (dark curve) for $B(n) = g^n$ (top) and $B(n) = kn + l$ (bottom), validating use of $\theta_0(n)$ in $z_0(n)$ for known functions $B(n)$ and showing sounding frequency of loopback FM can be accurately predicted.

It is clear that when B changes, so does the expression for θ_0, which might be seen as a limitation of this approach, except that it could be argued there are only a few ways in which one would expect B to change, and these can be expressed as functions with more subtle changes being accomplished via parameters settings. Furthermore, it is always possible to apply a numerical integration scheme if an analytical solution is not available.

Finally, it should also be noted that though it is possible to set a desired trajectory for sounding frequency $\omega_0(t)$ in $z_0(n)$, there is no guarantee this will be mappable to loopback FM parameters and the oscillator given by (50).

5. CONCLUSIONS

This work explored possible representations of the nonlinearly self-coupled oscillator, laying the groundwork for analysis and synthesis of systems with coupling in multiple modes. Beginning with the physical representation of the mass-spring system, an implementation of the oscillator using the bilinear transform is proposed, producing a biquadratic resonant filter that, without loss, is marginally stable and not well behaved when made time varying. Nevertheless, the assumed solution for equation of motion with frequency $\omega_0 + d_1\Re\{z_t(t)\}$, which could serve as an implementation when made discrete, is shown to be valid.

The modulation of the oscillator's frequency is formulated with a variant of FM synthesis called loopback FM, whereby the carrier oscillator loops back to serve as a modulator of its own frequency. Because of the integral relationship between frequency and phase, an alternate more numerically accurate closed-form representation of the system is required to produce an analytical solution to the oscillator's phase, ultimately revealing it's sounding frequency. This closed-form representation of the loopback FM oscillator is presented first in its static case, yielding mappings between the parameters of the two representations, and then in its more general form to allow for time-varying sounding frequency.

Acknowledgments

This work was strongly motivated by a conversation with Miller Puckette who had implemented a version of the system (5) in the real-time programming environment Pd. Inspired by the dynamics of the produced sound, the authors set out to describe it mathematically, with the hope the results could be used to enhance other synthesis models having similar feedback nonlinear characteristics. The authors are also sincerely grateful to the SMC reviewers whose in-depth comments and thorough reviews have greatly improved the presentation of this work.

Appendix A

The integral of $z_0(t)$ with respect to t may be represented as the sum of two integral terms:

$$\int_0^t z_0(t)\, dt = \int_0^t \frac{b_0}{1 + b_0 e^{j\omega_0 t}}\, dt + \int_0^t \frac{e^{j\omega_0 t}}{1 + b_0 e^{j\omega_0 t}}\, dt. \tag{75}$$

The first term of (75) may be represented by

$$\int_0^t \frac{b_0}{1 + b_0 e^{j\omega_0 t}}\, dt = b_0 \int_0^t \left(\frac{1 + b_0 e^{j\omega_0 t} - b_0 e^{j\omega_0 t}}{1 + b_0 e^{j\omega_0 t}} \right) dt, \tag{76}$$

which, when employing u-substitution where

$$u = 1 + b_0 e^{j\omega_0 t}, \frac{du}{dt} = j\omega_0 b_0 e^{j\omega_0 t}, dt = \frac{du}{j\omega_0 b_0 e^{j\omega_0 t}}, \tag{77}$$

may be further expressed as

$$\begin{aligned} \int_0^t \frac{b_0}{1 + b_0 e^{j\omega_0 t}}\, dt &= b_0 t - b_0 \int_0^t \frac{b_0 e^{j\omega_0 t}}{u} \frac{du}{j\omega_0 b_0 e^{j\omega_0 t}} \\ &= b_0 t - \frac{b_0}{j\omega_0} \int_0^u \frac{1}{u}\, du \\ &= b_0 t - \frac{b_0}{j\omega_0} \log(u). \end{aligned} \tag{78}$$

The second term of (75) is given by

$$\begin{aligned}\int_0^t \frac{e^{j\omega_0 t}}{1+b_0e^{j\omega_0 t}}\,dt &= \int_0^t \frac{e^{j\omega_0 t}}{u}\,\frac{du}{j\omega_0 b_0 e^{j\omega_0 t}} \\ &= \frac{1}{j\omega_0 b_0}\int_0^u \frac{1}{u}\,du \\ &= \frac{1}{j\omega_0 b_0}\log(u). \end{aligned} \quad (79)$$

Summing (78) and (79) and substituting values for u in (77) yields the final expression for the integral of $z_0(t)$:

$$\int_0^t z_0(t)\,dt = b_0 t + \frac{(1-b_0^2)\log\left(1+b_0e^{j\omega_0 t}\right)}{j\omega_0 b_0}. \quad (80)$$

Appendix B

The angle of an expression have the form

$$H(\theta) = \frac{c+e^{j\theta}}{1+ce^{j\theta}}, \quad (81)$$

is given by

$$\begin{aligned}\angle H(\theta) &= \angle\left(c+e^{j\theta}\right) - \angle\left(1+ce^{j\theta}\right) \\ &= \angle\left(e^{j\theta}\left(1+ce^{-j\theta}\right)\right) - \angle\left(1+ce^{j\theta}\right) \\ &= \angle e^{j\theta} + \angle\left(1+ce^{-j\theta}\right) - \angle\left(1+ce^{j\theta}\right) \\ &= \theta - 2\tan^{-1}\left(\frac{c\sin(\theta)}{1+c\cos(\theta)}\right).\end{aligned}$$

Appendix C

Employing u-substitution where $u=\sqrt{1-g^{2n}}$ and

$$\frac{du}{dn} = -\frac{1}{\not 2}(1-g^{2n})^{-1/2}\not 2 g^{2n}\log(g), \quad (82)$$

it follows that $-g^{2n} = u^2-1$ and

$$dn = -\frac{\sqrt{1-g^{2n}}}{g^{2n}\log(g)}\,du = \frac{u}{\log(g)(u^2-1)}\,du, \quad (83)$$

so that $\theta_0(n)$ in (71) may be expressed as

$$\theta_0(n) = \pm\int_0^n \omega_c T u\,dn = \frac{\omega_c T}{\log(g)}\int_0^u \frac{u^2}{u^2-1}\,du. \quad (84)$$

The integral term in (84) may be solved as

$$\begin{aligned}\int_0^u \frac{u^2}{u^2-1}du &= \int_0^u\left(1-\frac{1}{(1+u)(1-u)}\right)du \\ &= \int_0^u 1\,du - \int\left(\frac{(1+u)+(1-u)}{2(1+u)(1-u)}\right)du \\ &= u - \frac{1}{2}\int_0^u\left(\frac{1}{1-u}+\frac{1}{1+u}\right)du.\end{aligned}$$

Making substitutions

$$s = 1-u, ds=-du \quad \text{and} \quad p=1+u, dp=du \quad (85)$$

yields

$$\begin{aligned}\int_0^u \frac{u^2}{u^2-1}du &= u+\frac{1}{2}\int_0^s\frac{1}{s}\,ds - \frac{1}{2}\int_0^p \frac{1}{p}\,dp \\ &= u-\frac{1}{2}\left(\log(p)-\log(s)\right) \\ &= u - \frac{1}{2}\log\left(\frac{1+u}{1-u}\right)+C \\ &= u-\tanh^{-1}(u)+C,\end{aligned}$$

where C is the integration constant, and the final expression for $\theta_0(n)$ is

$$\theta_0(n) = \frac{\omega_c T}{\log(g)}\left(\sqrt{1-g^{2n}} - \tanh^{-1}\left(\sqrt{1-g^{2n}}\right)+C\right). \quad (86)$$

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